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METHODS OF SOLVING PROBLEMS WITH INTERVAL UNCERTAINTY

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Abstract. There can be uncertainty in initial data which is commonly interval. There are considered two kinds of optimization problems: interval linear programming (ILP), finding equilibrium position for interval von Neumann's model (bilinear problem). Definitions of different types of solutions and methods for finding these solutions are given. These methods imply reducing interval optimization problems to exact (ordinary) linear programming problems.

Keywords: interval uncertainty; interval linear programming; linear programming.

1. INTRODUCTION

Methods for solving linear programming (LP) problems were greatly developed in the 20th century due to the development of mathematical theory together with hardware and software [1]. But in fact optimal plans obtained from LP were often ineffective, not applicable. There were different factors that could lead to this ineffectiveness. One of them was inexactness of initial data [2].

For real-world linear economic models, numerical values of input matrices items are obtained using statistical data and expert estimates, therefore there can be an uncertainty, which is commonly interval. Using of average values may cause ineffectiveness of optimal solution, because uncertainty wasn't taken into account properly. Another approach is stochastic linear programming. This method requires that probability distributions for initial data are known while in practice this requirement does not hold in most cases.

.2. INTERVAL LINEAR PROGRAMMING PROBLEM

Let $\mathbf{A}$ be *interval matrix* with size $n \times m$

$$\mathbf{A} = [\underline{\mathbf{A}}; \overline{\mathbf{A}}] = [\mathbf{A}_c - \Delta; \mathbf{A}_c + \Delta],$$

where $\underline{\mathbf{A}}$ and $\overline{\mathbf{A}}$ are point matrices of interval lower and upper bounds of matrix $\mathbf{A}$; Δ is a point matrix,

$$\Delta = [\Delta_{i,j}]_{i=\overline{(1,n)},\ j=\overline{(1,m)}},\ \Delta_{ij} \geq 0;$$

$\mathbf{A}_c$ is matrix of *interval centers* $\mathbf{A}$ [4],

$$\mathbf{A}_c = (\overline{\mathbf{A}} + \underline{\mathbf{A}})/2.$$

Let us introduce interval vectors $\mathbf{b} = [\underline{\mathbf{b}}; \overline{\mathbf{b}}] = [\mathbf{b}_c - \delta; \mathbf{b}_c + \delta]$ with size $n \times 1$ and $\mathbf{c} = [\underline{\mathbf{c}}; \overline{\mathbf{c}}] = [\mathbf{c}_c - \gamma; \mathbf{c}_c + \gamma]$ with size $m \times 1$.

Hereinafter, parameters of productivity are meant by equilibrium position if it is not deaшned exactly.

Vector $x \in \mathbf{R}^m$ is a *weak solution* of system of interval linear equations

$$\mathbf{A}x = \mathbf{b},$$

if it satisfies $Ax = b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

Oettli-Prager Theorem. Vector $x \in \mathbf{R}^m$ is a weak solution of system $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$|\mathbf{A}_c x - \mathbf{b}_c| \leq \Delta |x| + \delta.$$

Checking weak solvability of linear interval equations system is NP-hard.

System of interval linear equations is *strongly solvable* if any system of point linear equations $Ax = b$ ($A \in \mathbf{A}$, $b \in \mathbf{b}$) is solvable. Checking strong solvability of linear interval equations system is NP-hard.

Vector $x \in \mathbf{R}^m$ is a *strong solution of system of interval linear equations* if x satisfies $Ax = b$ for any $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

Theorem [2]. Vector $x \in \mathbf{R}^m$ is a strong solution of system $\mathbf{A}x = \mathbf{b}$ if and only if x satisfies both inequalities:

$$\mathbf{A}_c x = \mathbf{b}_c,$$
$$\Delta |x| = \delta = 0.$$

Existence of strong solution is a rare case.

Vector $x \in \mathbf{R}^m$ is a weak solution of interval linear inequalities system

$$\mathbf{A}x \le \mathbf{b}$$

if x satisfies the system of point linear inequalities $Ax \le b$ for *some* $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

Gerlach Theorem [2]. Vector $x \in \mathbf{R}^m$ is a *weak solution of system* $\mathbf{A}x \le \mathbf{b}$ if and only if x satisfies

$$\mathbf{A}_c x - \Delta |x| \le \overline{\mathbf{b}}.$$

Vector $x \in \mathbf{R}^m$ is a *strong solution of interval linear inequalities system* if x satisfies the system of point linear inequalities $Ax \le b$ for <u>any</u> $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

System of interval linear equalities is *strongly solvable* if *any* system $Ax \le b$ is solvable for any matrices $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

It is proved [2] that if this system has feasible solution (x^1, x^2) then vector

$$x = x^1 - x^2$$

is a weak solution.

Checking strong solvability of linear interval inequalities system has polynomial complexity.

ILP problem is a family of point linear programming problems (LP problems)

$$\min\{c^T x \mid Ax = b,\ x \ge 0\}$$

for $A \in \mathbf{A}$, $b \in \mathbf{b}$ and $c \in \mathbf{c}$.

Let $f(A,b,c)$ be optimal solution of point LP problem with matrices (A,b,c).

Let

$$\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \inf\{ f(A,b,c) \mid A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}\}$$

be *lower bound of optimum for ILP* problem.

Let

$$\overline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \sup\{ f(A,b,c) \mid A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}\}$$

be *upper bound of optimum for ILP* problem.

Note that these bounds for optimum can be infinite.

Let us consider supplementary problem for upper bound using duality.

$$\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \sup\{(\mathbf{b}_c{}^T p + \delta^T |p| \mid \mathbf{A}_c p - \Delta^T |p| \le \overline{\mathbf{c}}\}.$$

Let $n \times 1$ vector y satisfy

$$y = \operatorname{sgn} p,$$

i. e. $y_i = \{-1;1\}$, $i = 1,2,\ldots,n$.

So there are 2^n combinations for vector y, the set of these combinations can be denoted Y^n. If vector y is fixed then we can solve LP subproblem for $\overline{\varphi}$

$$\varphi(y) = \max\{(\mathbf{b}_c{}^T p + \delta^T (y^T p) \mid \mathbf{A}_c p - \Delta^T (y^T p) \le \overline{\mathbf{c}}\}.$$

Value $\varphi(y)$ can be infinite. After using all combinations for vector y, we can calculate upper bound

$$\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c}) = \sup\{\varphi(y) \mid y \in Y^n\}.$$

To find lower bound of optimum the following problem should be solved

$$\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c}) = \min\{c^T x \mid \underline{\mathbf{A}}x \le \overline{\mathbf{b}},\ \overline{\mathbf{A}}x \ge \underline{\mathbf{b}},\ x \ge 0\}.$$

This problem is a point LP problem which can be solved for polynomial time.

Problems for $\overline{\varphi}$ and $\underline{f}$ can be split into series of subproblems, which can be solved using parallel computations with little exchange between processes.

Theorem [2]. For ILP problem he following statements are equivalent:

- for any matrices $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$ LP problem

  $$\max\{c^T x \mid Ax = b,\ x \ge 0\}$$

  has optimal solution;
- both lower bound and upper bound of optimum are finite;
- both lower bound and supplementary problem for upper bound are finite;
- system

  $$\{\ \overline{\mathbf{A}}^T p_1 - \underline{\mathbf{A}}^T p_2 \le \underline{c},\ \ p_1, p_2 \ge 0;\ \ p_1, p_2 \in \mathbf{R}^n\ \}$$

  is feasible and value $\overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})$ is finite.

The range of optimum is $[\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c}); \overline{\varphi}(\mathbf{A},\mathbf{b},\mathbf{c})]$ in every case.

The first step requires solving the only LP problem that is why it has polynomial complexity [2].

The second step requires solving of 2^n LP problems therefore it has exponential complexity (in the worst case).

Today LP problems can be effectively solved using parallel computations and exact rational-fractional calculations [3]. It is evident that each subproblem can be solved separately. So ILP problem has rather great potential for parallelism although coarse-grained parallelism is often used for each LP subproblem. CUDA C software engineering is suggested by author for parallel calculations. Another approach to solve ILP problem is interval simplex-method [4]. The idea is to use rules of interval arithmetics to calculate elements of simplex-table. But this method has a lot of restrictions and disadvantages, so it is rather controversial.

3. EQUILIBRIUM POSITION OF INTERVAL VON NEUMANN'S MODEL

A *general* equilibrium position for von Neumann's model (A,B), where A and B are given $n\times m$ input and output matrices with numerical nonnegative items,

$$a_{ij}, b_{ij} \geq 0,\ i=\overline{(1,n)},\ j=\overline{(1,m)};$$

is defined as a solution (λ, x, w) of the system of bilinear inequalities and equations

$$(A-\lambda B)x \leq 0,\ (x,e^m)=1,\ x\geq 0, \quad (1)$$

$$(A-\lambda B)^T w \geq 0,\ (w,e^n)=1,\ w\geq 0, \quad 2)$$

where

$$e^l \in R^l,\ e_i^l=1,\ i=\overline{(1,l)}.$$

A *non-degenerate* equilibrium position in the model under examination is an equilibrium position (λ, x, w) satisfying the additional condition $w^T Ax > 0$.

The extreme feasible values of λ can be found by solving the bilinear optimization problems

$$\lambda_n = \min\{\lambda \mid (A-\lambda B)x \leq 0, (x,e^m)=1, x\geq 0\}; \quad (3)$$

$$\lambda^* = \max\{\lambda \mid (A-\lambda B)^T w \geq 0, (w,e^n)=1, w\geq 0\}. \quad (4)$$

The numbers λ_n and $\lambda*$ are called von Neumann and Frobenius numbers of the model (A,B) respectively. The von Neumann number λ_n determines the maximum possible balanced growth rate, while the Frobenius number $\lambda*$ determines the minimum possible balanced growth rate and the workability of the model [7].

Vectors x, w of equilibrium position (λ, x, w) are called primal and dual von Neumann's rays corresponding the value of λ.

An *isolated pair* for von Neumann's model is a pair of arbitrary subsets

$$S \subset 1,2,\ldots,m$$

and

$$T \subset 1,2,\ldots,n,$$

for which if $j\in S$ and $i \notin T$ then

$$a_{ij} = b_{ij} = 0.$$

If there is no isolated pair in von Neumann's model then the von Neumann's number and the Frobenius number coincide [7].

Thus finding parameters of productivity (the Frobenius number $\lambda*$) and stable equilibrium position for von Neumann's model lies in solving the following *bilinear programming problem*

$$(\lambda^*, x^*, w^*) = \arg \max_{(\lambda,x,w)^T \in D(A,B)} \lambda$$

$$D(A,B) = \left\{ \begin{pmatrix} \lambda \\ x \\ w \end{pmatrix} \middle| \begin{array}{c} (A-\lambda B)x \leq 0, \\ (A-\lambda B)^T w \geq 0, \\ (x,e^m)=1, (w,e^n)=1, \\ \lambda\geq 0, x\geq 0, w\geq 0. \end{array} \right\} \quad (5)$$

Hereinafter, parameters of productivity are meant by equilibrium position if it is not defined exactly.

Numerical methods of solving this problem (5**)** are discussed in [82]. They are based on finding the roots of the monotone functions

$$u(\lambda) = \min_{x:(x,e^m)=1, x\geq 0} \ \max_{i=1,2,\ldots,n} \sum_{j=1}^{m} (a_{ij} - \lambda b_{ij}) x_j, \quad (6)$$

$$v(\lambda) = \max_{w:(w,e^n)=1, x\geq 0} \ \min_{j=1,2,\ldots,m} \sum_{i=1}^{n} (a_{ij} - \lambda b_{ij}) w_i. \quad (7)$$

When λ is fixed then the values of functions $u(\lambda)$ and $v(\lambda)$ equal to optimal values of mutually dual linear programming problems

$$\min\{u:\ (A-\lambda B)x \leq u, (x,e^m)=1, x\geq 0\}$$

and

$$\max\{v:\ (A-\lambda B)^T w \leq v, (w,e^n)=1, w\geq 0\}.$$

When λ is close to the roots of $u(\lambda), v(\lambda) \to 0$, problems (6) and (7) become degenerate because of appearance of zero basic variables u and v in optimal basis solution of this problems. That is why problems (6) and (7) cannot be solved with conventional means based on floating-point arithmetic.

Let us introduce *interval* von Neumann’s model by interval matrices of input $\mathbf{A} = \{[\underline{a_{ij}}, \overline{a_{ij}}]\}$ and output $\mathbf{B} = \{[\underline{b_{ij}}, \overline{b_{ij}}]\}$, $i=\overline{(1,n)}$, $j=\overline{(1,m)}$ [82, 6], which have matrices of centers

$$\mathbf{A}_c = (\underline{\mathbf{A}} + \overline{\mathbf{A}})/2,\ \mathbf{B}_c = (\underline{\mathbf{B}} + \overline{\mathbf{B}})/2.$$

The proves of theorems presented below are given in [8].

Theorem 1. Let triplet $(\underline{\lambda}, \underline{x}, \underline{w})$ be equilibrium position for von Neumann's model $(\underline{\mathbf{A}}, \overline{\mathbf{B}})$, and triplet $(\overline{\lambda}, \overline{x}, \overline{w})$ be equilibrium position for von Neumann's model $(\overline{\mathbf{A}}, \underline{\mathbf{B}})$; then the Frobenius number $\tilde{\lambda}$ for any point von Neumann's model $(\tilde{A}, \tilde{B})$: ($\tilde{A} \in \mathbf{A}$, $\tilde{B} \in \mathbf{B}$) belongs to $[\underline{\lambda}; \overline{\lambda}]$.

Strong solution (x_s, w_s) of interval model $(\mathbf{A}, \mathbf{B})$ provides equilibrium position $(\tilde{\lambda}, x_s, w_s)$ for any exact (point) von Neumann's model $(\tilde{A}, \tilde{B})$: ($\tilde{A} \in \mathbf{A}$, $\tilde{B} \in \mathbf{B}$).

Weak solution for interval von Neumann's model $(\mathbf{A}, \mathbf{B})$ is a pair of vectors (x', w'), under which set of constraints

$$\begin{cases} (\tilde{A} - \lambda \tilde{B}) x' \le 0; \\ (\tilde{A} - \lambda \tilde{B})^T w' \ge 0; \\ (x', e^m) = 1; \\ (w', e^n) = 1; \\ x', w', \lambda \ge 0 \end{cases}$$

is feasible for any exact von Neumann's model $(\tilde{A}, \tilde{B})$: ($\tilde{A} \in \mathbf{A}$, $\tilde{B} \in \mathbf{B}$).

Theorem 2. If set of constraints

$$\begin{cases} (\overline{\mathbf{A}} - \underline{\lambda} \underline{\mathbf{B}}) x'' \le 0; \\ (\underline{\mathbf{A}} - \overline{\lambda} \overline{\mathbf{B}})^T w'' \ge 0; \\ (x'', e^m) = 1; \\ (w'', e^n) = 1; \\ x'', w'' \ge 0 \end{cases}$$

is feasible under pair of vectors (x'', w'') then (x'', w'') is a weak solution of interval von Neumann's model $(\mathbf{A}, \mathbf{B})$.

Theorem 3. Let (x', w') be a weak solution for interval von Neumann's model $(\mathbf{A}, \mathbf{B})$. If exact von Neumann's model $(\tilde{A}, \tilde{B})$: ($\tilde{A} \in \mathbf{A}$, $\tilde{B} \in \mathbf{B}$) has

$$\lambda' = \max\{\lambda \mid (\tilde{A} - \lambda \tilde{B}) x' \le 0; (\tilde{A} - \lambda \tilde{B})^T w' \ge 0\}$$

then $\lambda' \in [\underline{\lambda_n}, \overline{\lambda}]$, where $\underline{\lambda_n}$ is the von Neumann's number for model $(\underline{\mathbf{A}}, \overline{\mathbf{B}})$.

Theorem 4. Let (x^*, w^*) be primal and dual Frobenius vectors for exact von Neumann's models $(\breve{A}, \breve{B})$ and $(\hat{A}, \hat{B})$, where

$$\hat{a}_{ij} - \breve{a}_{ij} = \Delta a_{ij} \ge 0, \; \hat{b}_{ij} - \breve{b}_{ij} = \Delta b_{ij} \ge 0.$$

Let the Frobenius number for model $(\breve{A}, \breve{B})$ is equal to $\breve{\lambda}$, and model $(\hat{A}, \hat{B})$ has the Frobenius number $\hat{\lambda}$, $\Delta\lambda = \hat{\lambda} - \breve{\lambda}$. Let model $(\breve{A}, \breve{B})$ has non-degenerate equilibrium position $(\breve{\lambda}, x^*, w^*)$. Then

$$\Delta\lambda = \frac{(w^*)^T (\Delta_A - \breve{\lambda} \Delta_B) x^*}{(w^*)^T (\breve{B} + \Delta_B) x^*}.$$

4. CONCLUSION

The main approach described is usage of lower and upper bounds to find different types of solutions. These solutions can be obtained from by solving series of exact LP problems by using parallel computations. It was shown that bilinear problem of finding equilibrium position for interval von Neumann's model can be treated similarly.

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